

# KÄHLER GROUPS AND DUALITY

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## CONTENTS

1. Introduction and results	1
2. Duality and finiteness	3
3. Cohomological properties of fundamental groups of smooth projective surfaces whose universal cover is Stein	7
4. On fundamental groups of smooth projective varieties whose universal cover is Stein : proof of theorem 3	8
5. Proof of theorem 1	9
6. Application to $p$ -adic lattices	12
Appendix A. What does conjecture 1 mean ?	13
Appendix B. Carlson-Toledo's conjecture and rigidity	15
References	16

## 1. INTRODUCTION AND RESULTS

**1.1. General setting.** The general setting for this paper is the study of topological properties of compact Kähler manifolds, in particular complex smooth projective varieties. The possible homotopy types for these spaces are essentially unknown (c.f. [28]). An *a priori* simpler question asks which finitely presented groups can be realized as fundamental groups of compact Kähler manifolds (the so-called Kähler groups).

It is well-known that any finitely presented group  $\Gamma$  can be realized as the fundamental group of a 4-dimensional compact real manifold, or even of a symplectic 4-manifold. A classical result of Serre [25] shows that any finite group can be realized as the fundamental group of a smooth complex projective variety. On the other hand there are many known obstructions for an infinite finitely presented group being Kähler (we refer to [1] for a panorama). Most of them come from Hodge theory (Abelian or not) *in cohomological degree one*.

As a prototype : let  $M$  be a compact Kähler manifold with fundamental group  $\Gamma$ . Classical Hodge theory shows the existence of a weight 1 pure Hodge structure on  $H^1(\Gamma, \mathbb{Q}) = H^1(M, \mathbb{Q})$ . Thus  $b^1(\Gamma)$  has to be even. By considering finite étale covers  $b^1(\Gamma')$  has to be even for any finite index subgroup  $\Gamma'$  of  $\Gamma$ . For example the free group on  $n$ -generator  $F_n$  is never Kähler,  $n \geq 1$ .

The most interesting conjecture concerning infinite Kähler groups, due to Carlson and Toledo and publicized by Kollar [19] and Simpson, deals with *cohomology in degree 2* :

**Conjecture 1** (Carlson-Toledo). *Let  $\Gamma$  be an infinite Kähler group. Then virtually  $b^2(\Gamma) > 0$ .*

*Remark 1.* Recall that a group  $\Gamma$  has virtually some property  $\mathcal{P}$  if a finite index subgroup  $\Gamma' \subset \Gamma$  has  $\mathcal{P}$ .

Conjecture 1 means (c.f. appendix A) that there exists a compact Kähler manifold  $M$  with  $\pi_1(M) = \Gamma'$  a finite index subgroup of  $\Gamma$  such that the rational Hurewicz morphism  $\pi_2(M) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_2(M, \mathbb{Q})$  is not surjective. This statement is highly non-trivial : there exists compact Kähler manifolds  $M$  (in fact smooth projective complex varieties) whose  $\pi_2(M)$  is very big, namely not finitely generated as a  $\mathbb{Z}\Gamma$ -module [9].

Conjecture 1 trivially holds true for fundamental groups of complex projective curves. By definition it is also satisfied by fundamental groups of compact Kähler hyperbolic manifolds [13]. The strongest evidence in its favor is that it holds true if the Kähler group  $\Gamma$  admits a finite dimensional complex representation  $\rho$  with  $H^1(\Gamma, \rho) \neq 0$  (c.f. theorem 10 appendix B). In particular it is true in all the known examples with very big  $\pi_2(M)$  (they satisfy  $b^1(\Gamma) > 0$ ).

**1.2. Results.** This paper is the first in a series of two studying cohomological properties of Kähler groups. It is essentially topological, using mainly duality in group cohomology and topological properties of Stein spaces. The second one [18] on the other hand is mainly geometric and uses non-Abelian Hodge theory. One will also consult [17] for a partial result towards conjecture 1.

Recall that a group  $\Gamma$  is an  $r$ -dimensional duality group (for some positive integer  $r$ ) if it satisfies a weak version of Poincaré duality : there exists a  $\mathbb{Z}\Gamma$ -module  $I$  such that for any  $\mathbb{Z}\Gamma$ -module  $A$  there is a natural isomorphism (i.e. induced by cap product with a fundamental class) :

$$\forall i \in \mathbb{N}, \quad H^i(\Gamma, A) \simeq H_{r-i}(\Gamma, I \otimes_{\mathbb{Z}} A) ,$$

where  $I \otimes_{\mathbb{Z}} A$  denotes the tensor product over  $\mathbb{Z}$  with diagonal action.

Many groups of geometric origin are duality groups : fundamental groups of aspherical manifolds, arithmetic lattices, mapping class groups ....

Our main result in this paper is the following :

**Theorem 1.** *Let  $\Gamma$  be an infinite linear  $r$ -dimensional duality group,  $r \geq 6$ . If  $\Gamma$  is a Kähler group then virtually*

$$b^2(\Gamma) + b^4(\Gamma) > 0 .$$

*Remark 2.* Requiring the group  $\Gamma$  to be linear is quite restrictive as we know that there exist many fundamental groups of smooth complex projective varieties which are not linear, not even residually finite [29]. This assumption is relaxed in [18] under some unboundedness condition.

Theorem 1 excludes many groups from being Kähler. A striking example is the following :

**Theorem 2.** *Let  $G_v$  be the group of  $F_v$ -points of an algebraic group  $\mathbf{G}_v$ , with reductive neutral component, over a non-Archimedean local field  $F_v$  of characteristic 0. Suppose that  $\text{rank}_{F_v} \mathbf{G}_v \geq 6$ . Then a cocompact lattice  $\Gamma \subset G_v$  is not Kähler.*

*Remark 3.* Notice that the conclusion of theorem 2 is predicted by Simpson's integrality conjecture under the weaker assumption  $\text{rank}_{F_v} \mathbf{G}_v \geq 2$  (c.f. section 6).

The proof of theorem 1 relies on the following ideas. First the group theoretical properties of the fundamental group of a space  $M$  are intimately linked to the geometric properties of the universal cover  $\tilde{M}$  of  $M$ . In the case of Kähler manifolds one has the following famous conjecture :

**Conjecture 2** (Shafarevich). *Let  $M$  be a connected compact Kähler manifold. Then its universal cover  $\tilde{M}$  is holomorphically convex.*

*Remark 4.* We recall that a complex space  $E$  is holomorphically convex if for any sequence  $(x_n)_{n \in \mathbb{N}}$  of points in  $E$  there exists a sequence of holomorphic functions  $(f_n \in \mathcal{O}_E(E))_{n \in \mathbb{N}}$  such that the complex sequence  $(f_n(x_n))_{n \in \mathbb{N}}$  is unbounded. Equivalently  $E$  admits a proper map  $E \rightarrow S$  with  $S$  Stein,  $S$  is called the Cartan-Remmert reduction of  $E$ .

From this point of view it is natural to first study the fundamental groups of compact Kähler manifolds  $M$  whose universal cover  $\tilde{M}$  is Stein. If one moreover assumes that  $M$  is a smooth projective complex variety one can assume that  $M$  is a smooth projective complex surface using Lefschetz's hyperplane theorem. Using the topological restrictions on  $\tilde{M}$  coming from the Stein condition, the Leray spectral sequence comparing the cohomologies of  $M$  and  $\Gamma$ , and the fact that  $\Gamma$  and  $M$  both satisfy duality but in different degrees, one shows the following :

**Theorem 3.** *Let  $\Gamma$  be an infinite oriented  $r$ -dimensional duality group,  $r \geq 6$ . Suppose  $\Gamma = \pi_1(M)$  where  $M$  is a connected smooth projective complex variety with Stein universal cover  $\tilde{M}$ . Then*

$$b^1(\Gamma) + b^4(\Gamma) > 0 .$$

*Remark 5.* Notice that in theorem 3 we are not asking for  $\Gamma$  to be linear. Notice also that in the conclusion of theorem 3 we do not need to pass to a finite index subgroup.

To deduce theorem 1 from theorem 3 one uses non-Abelian Hodge theory as in the proof of the Shafarevich conjecture for linear fundamental groups given in [16] and [10]. One is essentially reduced to proving a statement similar to theorem 3 with  $M$  only normal. Although I can't prove theorem 3 in the normal case the extra informations in the case at hand are enough to conclude the proof of theorem 1.

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**1.4. Notations.** If  $M$  is a topological space we abbreviate by  $H_\bullet(M)$  (resp.  $H^\bullet(M)$ ) its integral homology (resp. cohomology)  $H_\bullet(M, \mathbb{Z})$  (resp.  $H^\bullet(M, \mathbb{Z})$ ).

## 2. DUALITY AND FINITENESS

The goal of this section is to prove the following two lemmas (probably well-known to the algebraic topologist).

The first lemma implies that the duality requirement for  $\Gamma$  excludes the worse possible situation for conjecture 1 :

**Lemma 1.** *Let  $\Gamma$  be a duality group. Suppose  $\Gamma = \pi_1(M)$  for  $M$  a connected compact manifold. Then  $\pi_2(M)$  is a finitely generated (left)  $\mathbb{Z}\Gamma$ -module.*

The second lemma implies that given a closed oriented connected 4-manifold with fundamental group  $\Gamma$ , the  $\mathbb{Z}\Gamma$ -module  $\pi_2(M)$  satisfies a strong duality property as soon as  $\Gamma$  is an  $r$ -dimensional duality group,  $r > 3$  :

**Lemma 2.** *Let  $M$  be a closed oriented connected 4-manifold with fundamental group  $\Gamma$ . If  $H^2(\Gamma, \Lambda_{\mathbb{Z}}) = H^3(\Gamma, \Lambda_{\mathbb{Z}}) = 0$  then one has a natural isomorphism of (left)  $\Lambda_{\mathbb{Z}}$ -modules*

$$\pi_2(M) \simeq \overline{\text{Hom}_{\Lambda_{\mathbb{Z}}}(\pi_2(M), \Lambda_{\mathbb{Z}})}$$

(where  $\Lambda_{\mathbb{Z}}$  denotes the group ring  $\mathbb{Z}\Gamma$  and if  $R$  is a right  $\Lambda_{\mathbb{Z}}$ -module the notation  $\overline{R}$  denotes the associated left  $\Lambda_{\mathbb{Z}}$ -module).

**2.1. Finiteness conditions.** We first recall some basic definitions before proving lemmas 1 and 2.

**Definition 1.** A group  $G$  is said to be of type  $F_n$ ,  $0 \leq n < +\infty$ , if there is a  $K(G, 1)$  with a finite  $n$ -skeleton, i.e. with only finitely many cells in dimensions  $\leq n$ . We say that  $G$  is of type  $F_{\infty}$  if there is a  $K(G, 1)$  with all its skeleta finite. We say that  $G$  is of type  $F$  if there is a finite  $K(G, 1)$ .

Obviously any group of type  $F_0$ , a group  $G$  is of type  $F_1$  if and only if it is finitely generated and of type  $F_2$  if and only if it is finitely presented.

**Definition 2.** A group  $G$  is said to be of type  $FP_n$ ,  $0 \leq n < +\infty$ , if there is a projective resolution  $P_{\bullet}$  of  $\mathbb{Z}$  over  $\mathbb{Z}G$  such that  $P_i$  is finitely generated for  $i \leq n$ . We say that  $G$  is of type  $FP_{\infty}$  if there is a projective resolution  $P_{\bullet}$  of  $\mathbb{Z}$  over  $\mathbb{Z}G$  with  $P_i$  finitely generated for all  $i$ . We say that  $G$  is of type  $FP$  if there is a finite projective resolution  $P_{\bullet}$  of  $\mathbb{Z}$  over  $\mathbb{Z}G$ .

Obviously a group  $G$  of type  $F_n$  is of type  $FP_n$  as the cellular chain complex  $C_{\bullet}(\widetilde{K(G, 1)})$  (with  $\mathbb{Z}$ -coefficients) provides a free (thus projective) resolution of  $\mathbb{Z}$  as a  $\mathbb{Z}G$ -module up to degree  $n$ . One can show that  $G$  is of type  $F_1$  if and only if it is of type  $FP_1$  and that  $G$  is of type  $FP_2$  if and only if  $G = \tilde{G}/N$  where  $\tilde{G}$  is of type  $F_2$  and  $N$  is a perfect normal subgroup of  $\tilde{G}$ . In this case  $G$  is of type  $F_2$  if and only if  $N$  is finitely generated as a normal subgroup. Bestvina and Brady exhibited examples where this is not the case [2]. For higher  $n$  however a group  $G$  is of type  $F_n$ ,  $3 \leq n \leq \infty$ , if and only if  $G$  is finitely presented and of type  $FP_n$ .

**2.2. Duality groups.** If  $\Gamma$  is the fundamental group of an aspherical closed  $n$ -manifold  $M$  then it satisfies Poincaré duality :

$$H^i(\Gamma, A) \simeq H_{n-i}(\Gamma, I \otimes_{\mathbb{Z}} A)$$

where  $I$  denotes the orientation module of  $M$  and  $A$  can be any  $\mathbb{Z}\Gamma$ -module. This leads Bieri [4] to the following :

**Definition 3.** A group  $\Gamma$  is a Poincaré duality group of dimension  $n$  if there is a  $\mathbb{Z}\Gamma$ -module  $I$  (called the dualizing module of  $\Gamma$ ), which is isomorphic to  $\mathbb{Z}$  as a  $\mathbb{Z}$ -module, and a homology class  $\mu \in H_n(\Gamma, I)$  so that for any  $\mathbb{Z}\Gamma$ -module  $A$ , cap-product with  $\mu$  defines an isomorphism :

$$H^i(\Gamma, A) \simeq H_{n-i}(\Gamma, I \otimes_{\mathbb{Z}} A) .$$

Since the universal covering  $\tilde{M}^n$  of an aspherical manifold  $M^n$  is contractible it follows from Poincaré duality in the non-compact case that the cohomology with compact support of  $\tilde{M}^n$  is the same as that of  $\mathbb{R}^n$  i.e. :

$$H_c^i(\tilde{M}^n) = \begin{cases} 0, & \text{for } i \neq n, \\ \mathbb{Z}, & \text{for } i = n. \end{cases}$$

On the other hand  $\Gamma$  acts freely properly and cocompactly on the acyclic space  $\tilde{M}^n$  thus  $H^i(\Gamma, \mathbb{Z}\Gamma) \simeq H_c^i(\tilde{M}^n)$ . Johnson and Wall [14] proposed the following definition, which was shown to be equivalent to the previous one in [5] :

**Definition 4.** A group  $\Gamma$  is a Poincaré duality group of dimension  $n$  if  $\Gamma$  is of type  $FP$  and

$$H^i(\Gamma, \mathbb{Z}\Gamma) = \begin{cases} 0, & \text{for } i \neq n, \\ \mathbb{Z}, & \text{for } i = n. \end{cases}$$

The dualizing module is  $I = H^n(\Gamma, \mathbb{Z}\Gamma)$ . The main content of the equivalence of definitions 3 and 4 is that definition 3 forces  $\Gamma$  to be of type  $FP$ .

There are many interesting groups which satisfy definition 3 except for the requirement that  $I$  is isomorphic to  $\mathbb{Z}$  (as a  $\mathbb{Z}$ -module). This leads to the following theorem-definition [5] :

**Theorem 4** (Bieri-Eckmann). *A group  $\Gamma$  is a duality group of dimension  $n$  if it satisfies one the following equivalent conditions :*

- *There exists a  $\mathbb{Z}\Gamma$ -module  $I$  such that for any  $\mathbb{Z}\Gamma$ -module  $A$  there is an isomorphism induced by cap-product with a fundamental class :  $H^i(\Gamma, A) \simeq H_{n-i}(\Gamma, I \otimes_{\mathbb{Z}} A)$ .*
- *$\Gamma$  is of type  $FP$  and*

$$H^i(\Gamma, \mathbb{Z}\Gamma) = \begin{cases} 0, & \text{for } i \neq n, \\ I, & \text{for } i = n. \end{cases}$$

**2.3. A finiteness lemma.** Lemma 1 is an immediate corollary of the following more precise :

**Lemma 3.** *Let  $M$  be a connected CW-complex with fundamental group  $\Gamma$ . Let  $\Lambda_{\mathbb{Z}} = \mathbb{Z}\Gamma$  be the group ring of  $\Gamma$ .*

*Suppose  $M$  has a finite 3-skeleton.*

*Then  $\pi_2(M)$  is finitely generated as a  $\Lambda_{\mathbb{Z}}$ -module if and only if  $\pi_1(M)$  is of type  $FP_3$ .*

*Proof.* Suppose first that  $\pi_2(M)$  is finitely generated as a  $\Lambda_{\mathbb{Z}}$ -module. Thus one can construct the 3-skeleton of a  $K(\Gamma, 1)$  by adding finitely many 3-cells to the 3-skeleton  $M^{(3)}$  of  $M$ . As  $M^{(3)}$  is finite the group  $\Gamma$  is of type  $F_3$  thus also  $FP_3$ .

Conversely suppose  $\Gamma$  is of type  $FP_3$ . Thus there is a resolution  $(P_{\bullet}, d)$  of the  $\Lambda_{\mathbb{Z}}$ -module  $\mathbb{Z}$  with  $P_i$  a projective finitely generated  $\Lambda_{\mathbb{Z}}$ -module,  $i \leq 3$ . Let  $B_2 = d(P_3)$  the group of 2-boundaries. As  $P_3$  is  $\Lambda_{\mathbb{Z}}$ -finitely generated,  $B_2$  too and one has the following exact sequence of finitely generated  $\Lambda_{\mathbb{Z}}$ -modules :

$$(1) \quad 0 \longrightarrow B_2 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0 \quad .$$

On the other hand the cellular chain complex of the universal cover  $\tilde{M}$  gives the following exact sequence of  $\Lambda_{\mathbb{Z}}$ -modules :

$$(2) \quad 0 \longrightarrow Z_2(\tilde{M}) \longrightarrow C_2(\tilde{M}) \longrightarrow C_1(\tilde{M}) \longrightarrow C_0(\tilde{M}) \longrightarrow \mathbb{Z} \longrightarrow 0 \quad ,$$

where  $Z_2(\tilde{M}) = \ker(d_2 : C_2(\tilde{M}) \longrightarrow C_1(\tilde{M}))$ . As  $M$  has a finite 3-skeleton the  $C_i(\tilde{M})$ 's,  $0 \leq i \leq 3$ , are finitely generated  $\Lambda_{\mathbb{Z}}$ -modules.

Recall now the following classical

**Lemma 4** (Schanuel). *Let  $R$  be a ring. If*

$$0 \longrightarrow B_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

*and*

$$0 \longrightarrow B'_n \longrightarrow P'_{n-1} \longrightarrow \cdots \longrightarrow P'_1 \longrightarrow P'_0 \longrightarrow A \longrightarrow 0$$

*are exact sequences of  $R$ -modules with  $P_i, P'_i$  projective  $R$ -modules ( $0 \leq i \leq n-1$ ) then :*

$$B_n \oplus P'_{n-1} \oplus P_{n-2} \oplus \cdots \simeq B'_n \oplus P_{n-1} \oplus P'_{n-2} \oplus \cdots \quad .$$

Applying Schanuel's lemma to (1) and (2) one obtains :

$$B_2 \oplus C_2(\tilde{M}) \oplus P_1 \oplus \mathbb{Z} \simeq Z_2(\tilde{M}) \oplus P_2 \oplus C_1(\tilde{M}) \oplus P_0 \oplus \mathbb{Z} \quad .$$

This implies that  $Z_2(\tilde{M})$  is a finitely generated  $\Lambda_{\mathbb{Z}}$ -module, thus also its quotient  $\pi_2(M) = H_2(\tilde{M}) = Z_2(\tilde{M})/B_2(\tilde{M})$ .  $\square$

**2.4. Topology of closed 4-manifolds and proof of lemma 2.** Let  $M$  be a closed oriented connected 4-manifold with fundamental group  $\Gamma$  and universal cover  $\tilde{M}$ . Denote by  $\Lambda_{\mathbb{Z}} = \mathbb{Z}\Gamma$  the group ring of  $\Gamma$ . An interesting invariant is the equivariant intersection form :

$$s_M : \pi_2(M) \otimes_{\mathbb{Z}} \pi_2(M) \longrightarrow \Lambda_{\mathbb{Z}} ,$$

generalizing the well-known intersection product

$$H_2(M, \mathbb{Z}) \otimes_{\mathbb{Z}} H_2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

and defined as follows.

By Poincaré duality there is a canonical isomorphism  $H_c^i(\tilde{M}, \mathbb{Z}) \simeq H_{4-i}(\tilde{M}, \mathbb{Z})$  between cohomology with compact support and homology. In particular :

a) this induces an isomorphism

$$\varepsilon : H_c^4(\tilde{M}, \mathbb{Z}) \simeq H_0(\tilde{M}, \mathbb{Z}) = \mathbb{Z} .$$

b) Let  $h : \pi_2(M) \xrightarrow{\sim} H_2(\tilde{M}, \mathbb{Z})$  be the Hurewicz isomorphism. Using Poincaré duality we obtain a canonical isomorphism

$$\pi_2(M) \xrightarrow[\phi]{\sim} H_c^2(\tilde{M}, \mathbb{Z}) .$$

The cup-product on  $H_c^2(\tilde{M}, \mathbb{Z})$  is a  $\Gamma$ -invariant linear form with values in  $\mathbb{Z}$ . We thus define

$$s_M(x, y) = \sum_{g \in \Gamma} \varepsilon(\phi(g^{-1}x) \cup \phi(y)) \cdot g \in \Lambda_{\mathbb{Z}} .$$

Notice that this pairing is  $\Lambda_{\mathbb{Z}}$ -hermitian in the sense that for all  $\lambda \in \Lambda_{\mathbb{Z}}$  we have

$$s_M(\lambda \cdot x, y) = \lambda \cdot s_M(x, y) \quad \text{and} \quad s_M(y, x) = \overline{s_M(x, y)} ,$$

where  $\Lambda_{\mathbb{Z}}$  acts on itself by left translation and the involution  $\lambda \mapsto \bar{\lambda}$  on  $\Lambda_{\mathbb{Z}}$  is given by  $\bar{g} = g^{-1}$  for  $g \in \Gamma$ .

This pairing is controlled as follows. The Leray-Serre cohomological spectral sequence with coefficients in  $\Lambda_{\mathbb{Z}}$  for the classifying map  $M \longrightarrow K(\Gamma, 1)$ , whose homotopy fiber is  $\tilde{M}$ , yields the following short exact sequence of left  $\Lambda_{\mathbb{Z}}$ -modules :

$$0 \longrightarrow H^2(\Gamma, \Lambda_{\mathbb{Z}}) \longrightarrow H^2(M, \Lambda_{\mathbb{Z}}) \longrightarrow \overline{\text{Hom}_{\Lambda_{\mathbb{Z}}}(H_2(\tilde{M}, \Lambda_{\mathbb{Z}}), \Lambda_{\mathbb{Z}})} \longrightarrow H^3(\Gamma, \Lambda_{\mathbb{Z}}) \longrightarrow 0 .$$

By Poincaré duality for  $M$  and Hurewicz theorem we have an isomorphism of left  $\Lambda_{\mathbb{Z}}$ -modules :

$$H^2(M, \Lambda_{\mathbb{Z}}) \simeq H_2(M, \Lambda_{\mathbb{Z}}) \simeq H_2(\tilde{M}, \mathbb{Z}) \simeq \pi_2(M) .$$

Finally we get the following exact sequence of left  $\Lambda_{\mathbb{Z}}$ -modules :

$$(3) \quad 0 \longrightarrow H^2(\Gamma, \Lambda_{\mathbb{Z}}) \longrightarrow \pi_2(M) \xrightarrow{\tau_M} \overline{\text{Hom}_{\Lambda_{\mathbb{Z}}}(\pi_2(M), \Lambda_{\mathbb{Z}})} \longrightarrow H^3(\Gamma, \Lambda_{\mathbb{Z}}) \longrightarrow 0 .$$

One easily checks that the map  $\tau_M$  is the natural map associated to the pairing  $s_M$ . In particular the kernel of  $s_M$  is nothing else than  $H^2(\Gamma, \Lambda_{\mathbb{Z}})$ , its cokernel is  $H^3(\Gamma, \Lambda_{\mathbb{Z}})$ .

From this short discussion we immediately get lemma 2.

### 3. COHOMOLOGICAL PROPERTIES OF FUNDAMENTAL GROUPS OF SMOOTH PROJECTIVE SURFACES WHOSE UNIVERSAL COVER IS STEIN

**Proposition 1.** *Let  $M$  be a connected 4-dimensional CW-complex with universal cover  $\tilde{M}$  and fundamental group  $\Gamma$ . Let  $R$  be any (left)  $\Gamma$ -module. If  $H^3(\tilde{M}) = H^4(\tilde{M}) = 0$  then*

(a) *the following exact sequence of  $\Gamma$ -modules holds :*

$$0 \longrightarrow H^2(\Gamma, R) \longrightarrow H^2(M, R) \longrightarrow (H^2(\tilde{M}) \otimes_{\mathbb{Z}} R)^{\Gamma} \longrightarrow H^3(\Gamma, R) \longrightarrow H^3(M, R) \longrightarrow H^1(\Gamma, H^2(\tilde{M}) \otimes_{\mathbb{Z}} R) \longrightarrow H^4(\Gamma, R) \longrightarrow H^4(M, R) \longrightarrow H^2(\Gamma, H^2(\tilde{M}) \otimes_{\mathbb{Z}} R) \longrightarrow H^5(\Gamma, R) \longrightarrow 0 .$$

(b)  $\forall i \geq 3, \quad H^i(\Gamma, H^2(\tilde{M}) \otimes_{\mathbb{Z}} R) \simeq H^{i+3}(\Gamma, R) .$

*Proof.* Let us write the Leray-Serre spectral sequence associated to the classifying map  $f : M \longrightarrow K(\Gamma, 1)$  (whose homotopy fiber is  $\tilde{M}$ ) for the local coefficients  $R$  :

$$H^p(\Gamma, H^q(\tilde{M}) \otimes_{\mathbb{Z}} R) \Longrightarrow H^{p+q}(M, R) .$$

As  $\tilde{M}$  is simply connected, the  $E_2$ -page of this spectral sequence has only two non-zero lines :

$$E_2^{p,0} = H^p(\Gamma, R) \quad \text{and} \quad E_2^{p,2} = H^p(\Gamma, H^2(\tilde{M}) \otimes_{\mathbb{Z}} R) .$$

It implies that  $E_3 = E_2$  and then  $E_4 = E_5 = \dots = E_{\infty}$ . Thus :

$$\begin{aligned} E_{\infty}^{0,0} &= R^{\Gamma} \\ E_{\infty}^{0,1} &= 0, \quad E_{\infty}^{1,0} = H^1(\Gamma, R), \\ E_{\infty}^{0,2} &= \ker((H^2(\tilde{M}) \otimes_{\mathbb{Z}} R)^{\Gamma} \xrightarrow{d_3} H^3(\Gamma, R)), \quad E_{\infty}^{1,1} = 0, \quad E_{\infty}^{2,0} = H^2(\Gamma, R), \end{aligned}$$

and for all  $i, j, i+j \geq 3$ ,

$$E_{\infty}^{i,j} = \begin{cases} \ker(H^i(\Gamma, H^2(\tilde{M}) \otimes_{\mathbb{Z}} R) \xrightarrow{d_3} H^{i+3}(\Gamma, R)) & \text{if } j = 2, \\ H^i(\Gamma, R)/d_3(H^{i-3}(\Gamma, H^2(\tilde{M}) \otimes_{\mathbb{Z}} R)) & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

From this one deduces the following short exact sequences :

$$0 \longrightarrow H^2(\Gamma, R) \longrightarrow H^2(M, R) \longrightarrow \ker((H^2(\tilde{M}) \otimes_{\mathbb{Z}} R)^{\Gamma} \xrightarrow{d_3} H^3(\Gamma, R)) \longrightarrow 0 ,$$

and for  $i \geq 3$  :

$$0 \longrightarrow \frac{H^i(\Gamma, R)}{d_3(H^{i-3}(\Gamma, H^2(\tilde{M}) \otimes_{\mathbb{Z}} R))} \longrightarrow H^i(M, \mathbb{C}) \longrightarrow \ker(H^{i-2}(\Gamma, H^2(\tilde{M}) \otimes_{\mathbb{Z}} R) \xrightarrow{d_3} H^{i+1}(\Gamma, R)) \longrightarrow 0 .$$

The proposition then follows from the consideration of these short exact sequences for  $i \leq 5$  and the fact that  $H^i(M, \cdot) = 0$  for  $i \geq 5$ .  $\square$

Studying similarly the Leray-Serre homology spectral sequence one obtains the dual statements :

**Proposition 2.** *Let  $M$  be a connected 4-dimensional CW-complex with universal cover  $\tilde{M}$  and fundamental group  $\Gamma$ . Let  $R$  be any (left)  $\Gamma$ -module. If  $H_3(\tilde{M}) = H_4(\tilde{M}) = 0$  then :*

(a) *the following exact sequence of  $\Gamma$ -modules holds :*

$$0 \longleftarrow H_2(\Gamma, R) \longleftarrow H_2(M, R) \longleftarrow (H_2(\tilde{M}) \otimes_{\mathbb{Z}} R)_{\Gamma} \longleftarrow H_3(\Gamma, R) \longleftarrow H_3(M, R) \longleftarrow H_1(\Gamma, H_2(\tilde{M}) \otimes_{\mathbb{Z}} R) \longleftarrow H_4(\Gamma, R) \longleftarrow H_4(M, R) \longleftarrow H_2(\Gamma, H_2(\tilde{M}) \otimes_{\mathbb{Z}} R) \longleftarrow H_5(\Gamma, R) \longleftarrow 0 .$$

(b)  $\forall i \geq 3, \quad H_i(\Gamma, H_2(\tilde{M}) \otimes_{\mathbb{Z}} R) \simeq H_{i+3}(\Gamma, R) .$

**Corollary 1.** *Let  $M$  be a connected complex surface (not necessarily smooth) with Stein universal cover  $\tilde{M}$  and fundamental group  $\Gamma$ . Let  $R$  be any (left)  $\Gamma$ -module. Then the conclusions of proposition 1 and proposition 2 hold true.*

*Proof.* The fact that a Stein surface  $\tilde{M}$  (not necessarily smooth) satisfies  $H^i(\tilde{M}) = 0$ ,  $i \geq 3$ , is a classical result of Narasimhan [23]. In fact such an  $\tilde{M}$  is homotopy equivalent to a 2-dimensional CW-complex [12].  $\square$

#### 4. ON FUNDAMENTAL GROUPS OF SMOOTH PROJECTIVE VARIETIES WHOSE UNIVERSAL COVER IS STEIN : PROOF OF THEOREM 3

##### 4.1. A vanishing lemma.

**Lemma 5.** *Let  $\Gamma$  be an infinite group of type  $FP_3$  satisfying*

$$H^2(\Gamma, \Lambda) = H^3(\Gamma, \Lambda) = 0 \quad ,$$

*where  $\Lambda = \mathbb{Q}\Gamma$ . Suppose  $\Gamma$  is the fundamental group of a connected CW-complex  $M$ . Then*

$$(H^2(\tilde{M})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \pi_2(M)_{\mathbb{Q}})^{\Gamma} = 0$$

*(for the diagonal action of  $\Gamma$  on  $H^2(\tilde{M}) \otimes_{\mathbb{Z}} \pi_2(M)$ ).*

*Proof.* By lemma 3  $\pi_2(M)$  is a finitely generated (left)  $\mathbb{Z}\Gamma$ -module. Thus there exists a positive integer  $i$  and a surjective morphism of (left)  $\Lambda$ -modules :

$$\Lambda^i \twoheadrightarrow \pi_2(M)_{\mathbb{Q}} \quad .$$

This implies that one has an injective morphism of (left)  $\Lambda$ -modules :

$$\overline{\text{Hom}_{\Lambda}(\pi_2(M)_{\mathbb{Q}}, \Lambda)} \hookrightarrow \Lambda^i \quad .$$

By lemma 2 :

$$\pi_2(M) \simeq \overline{\text{Hom}_{\Lambda_{\mathbb{Z}}}(\pi_2(M), \Lambda_{\mathbb{Z}})} \quad .$$

Finally  $H^2(\tilde{M})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \pi_2(M)_{\mathbb{Q}} \subset H^2(\tilde{M})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda^i$  and one is reduced to proving : for any non-trivial (left)  $\Lambda$ -module  $R$  one has

$$(R \otimes_{\mathbb{Q}} \Lambda)^{\Gamma} = 0 \quad .$$

This follows from the fact that the infinite group  $\Gamma$  has only infinite orbits in its action by left translation on  $\Lambda = \mathbb{Q}\Gamma$ .  $\square$

##### 4.2. The topological theorem.

**Theorem 5.** *Let  $\Gamma$  be an infinite oriented  $r$ -dimensional duality group,  $r \geq 6$ . Suppose  $\Gamma = \pi_1(M)$  where  $M$  is a connected oriented compact  $\mathcal{C}^{\infty}$  4-manifold whose universal cover  $\tilde{M}$  has the homotopy type of a 2-dimensional CW-complex. Then*

$$b^1(\Gamma) + b^4(\Gamma) > 0 \quad .$$

*Proof.* Assume by contradiction  $b^1(\Gamma) + b^4(\Gamma) = 0$ . As  $b^1(M) = b^1(\Gamma)$  and  $b^3(M) = b^1(M)$  by Poincaré duality for  $M$ , one obtains  $b^3(M) = 0$ .

The homological exact sequence (proposition 2, (a)) gives the exact sequence :

$$H_4(\Gamma)_{\mathbb{Q}} \longrightarrow H_1(\Gamma, \pi_2(M)_{\mathbb{Q}}) \longrightarrow H_3(M)_{\mathbb{Q}}$$



from which we deduce :

$$(4) \quad H_1(\Gamma, \pi_2(M)_{\mathbb{Q}}) = 0 \quad .$$

By Poincaré duality for  $M$  one deduces from equation (4) :

$$(5) \quad H^3(M, \pi_2(M)_{\mathbb{Q}}) \simeq H_1(M, \pi_2(M)_{\mathbb{Q}}) = 0 \quad .$$

Let us now consider the cohomological exact sequence (proposition 1, (a)) with coefficients  $\pi_2(M)_{\mathbb{Q}}$  :

$$(6) \quad (H^2(\tilde{M})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \pi_2(M)_{\mathbb{Q}})^{\Gamma} \longrightarrow H^3(\Gamma, \pi_2(M)_{\mathbb{Q}}) \longrightarrow H^3(M, \pi_2(M)_{\mathbb{Q}}) \quad .$$

As  $\Gamma$  is an  $r$ -dimensional duality group with  $r \geq 3$  :

$$H^2(\Gamma, \Lambda) = H^3(\Gamma, \Lambda) = 0 \quad .$$

By lemma 5  $(H^2(\tilde{M})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \pi_2(M)_{\mathbb{Q}})^{\Gamma} = 0$ . On the other hand by equation (5) one gets  $H^3(M, \pi_2(M)_{\mathbb{Q}}) = 0$ . We deduce from equation (6) :

$$(7) \quad H^3(\Gamma, \pi_2(M)_{\mathbb{Q}}) = 0 \quad .$$

However by duality for  $\Gamma$  :

$$H^3(\Gamma, \pi_2(M)_{\mathbb{Q}}) \simeq H_{r-3}(\Gamma, I \otimes_{\mathbb{Z}} \pi_2(M)_{\mathbb{Q}}) \quad ,$$

and by the homological isomorphism (proposition 2, (b)) one obtains as  $r \geq 6$  :

$$H_{r-3}(\Gamma, I \otimes \pi_2(M)_{\mathbb{Q}}) \simeq H_r(\Gamma, I) \simeq H^0(\Gamma, \mathbb{Q}) \simeq \mathbb{Q} \quad .$$

Finally  $H^3(\Gamma, \pi_2(M)_{\mathbb{Q}}) \simeq \mathbb{Q}$ , contradiction to equation (7).  $\square$

**4.3. Proof of theorem 3.** Let  $\Gamma$  be a group as in theorem 3. Thus  $\Gamma = \pi_1(M)$ ,  $M$  complex smooth projective variety of (complex) dimension  $n$  with  $\tilde{M}$  Stein. If  $n = 1$  the variety  $M$  is a smooth projective curve, of genus  $\geq 1$  as  $\Gamma$  is infinite. In particular  $M$  is a  $K(\Gamma, 1)$  and  $b^1(\Gamma) = b^1(M) > 0$  thus the conclusion of theorem 1 is valid in this case. Therefore we can assume  $n \geq 2$ .

Fix  $\mathcal{O}_M(1)$  an ample line bundle on  $M$ . By Lefschetz hyperplane theorem any smooth surface  $N$  complete intersection in  $M$  of hyperplane sections corresponding to  $\mathcal{O}_M(1)$  still has fundamental group  $\Gamma$ . Moreover the universal cover  $\tilde{N}$  of such an  $N$  is an irreducible component of the preimage of  $N$  in  $\tilde{M}$ . Thus  $\tilde{N}$  is a closed analytic submanifold of a Stein manifold, hence a Stein manifold. Replacing  $M$  by  $N$  we can assume  $n = 2$ .

Finally  $M$  satisfies the hypotheses of theorem 5 and we conclude.  $\square$

## 5. PROOF OF THEOREM 1

**5.1. Shafarevich maps.** In [10, theor.2] Eyssidieux shows the following

**Theorem 6** (Eyssidieux). *Let  $X$  be a (connected) smooth complex projective variety. Let  $\rho : \pi_1(X) \longrightarrow GL(n, \mathbb{C})$  be a semi-simple representation.*

*There exists a diagram of analytic morphisms*

$$\begin{array}{ccc} \tilde{X} / \ker \rho & \xrightarrow{\widetilde{\text{sh}}_{\rho}} & \widetilde{S}_{\rho}(X) \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{\text{sh}_{\rho}} & \text{sh}_{\rho}(X) \end{array}$$

where :

- $\text{sh}_\rho(X)$  is a normal projective variety, the morphism  $\text{sh}_\rho : X \rightarrow \text{sh}_\rho(X)$  has connected fibers and for any morphism  $Z \rightarrow X$ , with  $Z$  a smooth connected projective variety, the image  $\text{sh}_\rho(Z)$  is a point if and only if  $\rho(\pi_1(Z))$  is finite.
- $\widetilde{S}_\rho(X)$  is a normal analytic variety without compact positive dimensional analytic subspaces with a proper discontinuous action of  $\rho(\Gamma)$ , the morphism  $\tilde{X}/\ker \rho \rightarrow \widetilde{S}_\rho$  is  $\Gamma$ -equivariant, proper and satisfies  $\text{sh}_\rho(X) = \widetilde{S}_\rho(X)/\rho(\Gamma)$ .

In the special case where  $\rho$  is rigid Eyssidieux proves the holomorphic convexity of the cover  $\tilde{X}/\ker \rho$  (c.f. [10, theor.3] and [10, section 4] for the proof) :

**Theorem 7** (Eyssidieux). *With the notations and assumptions of theorem 6 suppose moreover that  $\rho$  is rigid. Then  $\widetilde{S}_\rho(X)$  is Stein (and thus  $\tilde{X}/\ker \rho$  is holomorphically convex).*

*Remarks 1.* (a) In addition to [10, theor.2] the reader might want to look at [10, theor.2.1.7] and [10, prop.2.2.20] which deal with the case of a rigid  $\rho$  (the only case of interest for us).  
 (b) In the case where  $X$  is a surface, theorem 6 and theorem 7 in this context are proven in [16]. However we refer to [10] for a more detailed construction.  
 (c) Of course the results of [10] are more general. In particular the holomorphic convexity of  $\tilde{X}/\ker \rho$  generalizes to the case where one replaces  $\rho$  by a quasi compact absolutely constructible set  $R$  of conjugacy classes of semisimple representations in the sense of Simpson and  $\tilde{X}/\ker \rho$  by  $\tilde{X}_R$  the cover of  $X$  corresponding to the intersection of their kernels.

We will need one more fact from [10]. Suppose  $\rho$  is a rigid representation. Thus one can assume that  $\rho$  takes values in  $\mathbf{GL}(n, K)$ ,  $K$  number field. Let  $\mathbf{G}$  be the natural  $K$ -form of the reductive group Zariski closure of  $\rho(\Gamma)$  in  $\mathbf{GL}(n, \mathbb{C})$ . As  $\Gamma$  is of finite type one has  $\rho(\Gamma) \subset \mathbf{G}((\mathcal{O}_K)_S)$  where  $\mathcal{O}_K$  denotes the ring of integers of  $K$  and  $S$  is a finite set of places of  $K$ . We assume  $S$  minimal i.e. for any  $v \in S$  the representation  $\rho_v : \Gamma \rightarrow \mathbf{G}(K_v)$  does not have bounded image. Let  $S_{\text{ar}} \subset S$  the set of archimedean places and  $S_f = S \setminus S_{\text{ar}}$  the set of finite places. As in [10, p.524] let

$$h : \tilde{X}/\ker \rho \rightarrow R_S := \prod_{v \in S_f} \Delta_v \times \prod_{v \in S_{\text{ar}}} R_v$$

be the natural  $\Gamma$ -equivariant harmonic map to the product of the Bruhat-Tits buildings  $\Delta_v$  associated to the  $p$ -adic groups  $\mathbf{G}_{K_v}$ ,  $v \in S_f$ , and of the symmetric spaces  $R_v$  associated to the real Lie group  $\mathbf{G}_{K_v}$ ,  $v \in S_{\text{ar}}$ . Then by construction of  $\widetilde{S}_\rho(X)$  one has (c.f. [10, p.524]) :

**Lemma 6** (Eyssidieux). *The harmonic map  $h$  factorizes (equivariantly) through  $\widetilde{S}_\rho(X)$  :*

$$\begin{array}{ccc} \tilde{X}/\ker \rho & \xrightarrow{h} & R_S \\ \text{sh}_\rho \downarrow & \nearrow & \\ \widetilde{S}_\rho(X) & & \end{array}$$

5.1.1. *Proof of theorem 1.* Let  $\Gamma$  be a group as in theorem 1. Thus  $\Gamma = \pi_1(X)$ ,  $X$  compact Kähler manifold. Let  $\rho : \Gamma \hookrightarrow \mathbf{GL}(n, \mathbb{C})$  be a linear embedding.

Suppose by contradiction  $b^2(\Gamma) + b^4(\Gamma) = 0$ . As explained in the appendix B our assumption  $b^2(\Gamma) = 0$  implies that  $\Gamma$  is *schematically rigid*, meaning that for any  $n \in \mathbb{N}$  and any representation  $\rho : \Gamma \rightarrow \mathbf{GL}(n, \mathbb{C})$  one has  $H^1(\Gamma, \text{Ad}\rho) = 0$ . Considering the trivial representation we obtain in

particular that

$$(8) \quad b^1(\Gamma) = 0 \quad .$$

Schematic rigidity implies that  $\Gamma$  is *reductive*, meaning that all its linear finite-dimensional complex representations are semi-simple (c.f. [17, section 3.5]). In particular the Zariski closure  $\mathbf{G}$  of  $\Gamma$  in  $\mathbf{GL}(n, \mathbb{C})$  is reductive. As  $b^1(\Gamma) = 0$  one can assume that  $\mathbf{G}$  is simple.

**Lemma 7.** *The group  $\Gamma$  is virtually the fundamental group of a smooth projective variety.*

*Proof.* This follows from [32, theor.1] generalizing [22, Main Theorem], and the faithfulness of  $\rho$ , as follows.

By [32, theor.1.(b)] there exists an analytic morphism with connected fibers  $l : X \rightarrow M$ ,  $M$  normal irreducible projective variety, such that  $\rho$  factorizes through  $\pi_1(M)$ . Let  $\sigma_M : \hat{M} \rightarrow M$  be a resolution of singularities of  $M$ . One can find a modification  $\sigma_X : \hat{X} \rightarrow X$  and a map  $\hat{l} : \hat{X} \rightarrow \hat{M}$  with connected fibers such that the diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{l}} & \hat{M} \\ \sigma_X \downarrow & & \downarrow \sigma_M \\ X & \xrightarrow{l} & M \end{array}$$

commutes. Taking fundamental groups one obtain the commutative diagram of groups :

$$(9) \quad \begin{array}{ccc} \pi_1(\hat{X}) & \xrightarrow{\hat{l}_*} & \pi_1(\hat{M}) \\ (\sigma_X)_* \downarrow & & \downarrow (\sigma_M)_* \\ \pi_1(X) & \xrightarrow{l_*} & \pi_1(M) \end{array} \quad .$$

As  $\sigma$  is a modification and  $X$  is smooth the induced map  $(\sigma_X)_* : \pi_1(\hat{X}) \rightarrow \pi_1(X) = \Gamma$  is an isomorphism. As  $\hat{l}$  and  $l$  are connected the morphisms  $\hat{l}_*$  and  $l_*$  are surjective. As  $\rho$  is faithful and factorizes through  $\pi_1(M)$  the diagram (9) implies that  $\Gamma = \pi_1(\hat{M})$  and the result.  $\square$

Replacing  $X$  by  $\hat{M}$  one can assume that  $X$  is a smooth connected projective complex variety of complex dimension  $n$ . If  $n = 1$  the variety  $X$  is a smooth curve, of genus  $\geq 1$  as  $\Gamma$  is infinite. In particular  $X$  is a  $K(\Gamma, 1)$  and  $b^2(\Gamma) = b^2(X) = 1$  thus the conclusion of theorem 1 is valid in this case. Thus we can assume  $n \geq 2$ .

Once more one can assume that  $X$  is as surface ( $n = 2$ ) by replacing  $X$  by a sufficiently ample smooth complete intersection surface in  $X$ . By theorem 6 we have a diagram of Shafarevich maps

$$\begin{array}{ccc} \tilde{X} / \ker \rho & \xrightarrow{\widetilde{\text{sh}}_\rho} & \widetilde{S}_\rho(X) \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{\text{sh}_\rho} & \text{sh}_\rho(X) \end{array} \quad .$$

Moreover by theorem 7 the normal space  $\widetilde{S}_\rho(X)$  is Stein.

**Lemma 8.** *The map  $\rho : \Gamma \rightarrow \mathbf{G}((\mathcal{O}_K)_S)$  factorizes through  $\pi_1(\text{sh}_\rho(X))$ .*

*Proof.* By lemma 6 the harmonic map  $h$  factorizes (equivariantly) through  $\tilde{S}_\rho(X)$  :

$$\begin{array}{ccc} \tilde{X}/\ker \rho & \xrightarrow{h} & R_S \\ \text{sh}_\rho \downarrow & \nearrow & \\ \tilde{S}_\rho(X) & & \end{array}$$

The group  $\Gamma$  acts properly discontinuously on  $R_S$ . By replacing  $\Gamma$  if necessary by a finite index subgroup we can assume that  $\Gamma$  acts freely. Then the previous diagram induces a factorization :

$$\begin{array}{ccc} X & \xrightarrow{h} & R_S/\Gamma \\ \text{sh}_\rho \downarrow & \nearrow & \\ \text{sh}_\rho(X) & & \end{array}$$

At the level of  $\pi_1$  it gives a diagram

$$(10) \quad \begin{array}{ccc} \Gamma & \xrightarrow{h_* = \text{Id}} & \Gamma \\ \text{sh}_{\rho*} \downarrow & \nearrow & \\ \pi_1(\text{sh}_\rho(X)) & & \end{array}$$

and the result.  $\square$

**Lemma 9.** *Under the assumptions of theorem 1 the map  $\text{sh}_\rho : X \rightarrow \text{sh}_\rho(X)$  is a modification and the morphism  $(\text{sh}_\rho)_* : \Gamma \rightarrow \pi_1(\text{sh}_\rho(X))$  is an isomorphism.*

*Proof.* By definition the map  $\text{sh}_\rho$  is surjective thus  $\text{sh}_\rho(X)$  is of (complex) dimension at most 2. By the previous lemma the representation  $\rho$  factorises through  $\pi_1(\tilde{S}_\rho)$ . As  $\Gamma = \rho(\Gamma)$  is infinite the group  $\pi_1(\tilde{S}_\rho)$  too. Thus  $\text{sh}_\rho(X)$  is not a point nor  $\mathbf{P}^1\mathbb{C}$ . If  $\text{sh}_\rho(X)$  is a (smooth) curve of genus  $g \geq 1$  then  $b^1(X) \geq b^1(\text{sh}_\rho(X)) > 0$ , contradiction to equality (8). Finally  $\text{sh}_\rho(X)$  is of dimension 2. By definition the fibers of  $\text{sh}_\rho$  are connected thus  $\text{sh}_\rho : X \rightarrow \text{sh}_\rho(X)$  is a modification.

As  $\text{sh}_\rho : X \rightarrow \text{sh}_\rho(X)$  is a modification and  $\text{sh}_\rho(X)$  is normal we obtain that the map  $\text{sh}_{\rho*} : \Gamma \rightarrow \pi_1(\text{sh}_\rho(X))$  is surjective. By considering the diagram (10) we obtain that this map is an isomorphism.  $\square$

As  $p : X \rightarrow M := \text{sh}_\rho(X)$  is a modification inducing an isomorphism on fundamental groups the universal cover  $\tilde{X}$  is also a modification of  $\tilde{M}$ . Thus the map  $\tilde{p} : \tilde{X} \rightarrow \tilde{M}$  can be realized topologically as a cofibration with cofiber  $C_{\tilde{p}}$  a union of suspension of complex curves. As  $\tilde{M}$  is a Stein surface (possibly singular) it has the homotopy type of a 2-dimensional CW-complex. Writing the (co)homology long exact sequence for the pair  $(M, X)$  we obtain that  $X$  does not have any (co)homology in degree larger than 2. Thus  $X$  still satisfies the hypotheses of theorem 5, which implies the result.  $\square$

## 6. APPLICATION TO $p$ -ADIC LATTICES

**6.1. Proof of theorem 2.** More generally one can prove the following

**Theorem 8.** *Let  $G = \prod_{v \in S} G_v$ , where  $S$  denotes a finite set and  $G_v$ ,  $v \in S$ , is the group of  $F_v$ -points of an algebraic group  $\mathbf{G}_v$ , with reductive neutral component, over a non-archimedean local field  $F_v$  of characteristic 0. Suppose that  $d(G) := \sum_{v \in S} \text{rk}_{F_v} \mathbf{G}_v$  satisfies  $d(G) \geq 6$ .*

*Then an irreducible cocompact lattice  $\Gamma \subset G$  is never Kähler.*

*Proof.* Clearly the group  $\Gamma$  is linear. In particular  $\Gamma$  admits a finite index subgroup which is torsion-free. As a group is Kähler if and only if any finite index subgroup of  $\Gamma$  is Kähler, we can assume that  $\Gamma$  is torsion-free.

By theorem [6, theor. 6.2]  $\Gamma$  is a duality group of dimension  $d(G)$ .

Moreover by a famous result of Garland [11] (under some restriction on the residual characteristic of the  $F_v$ 's, extended by Casselman [8] to the general case) any finite index subgroup  $\Gamma'$  of  $\Gamma$  satisfies

$$H^i(\Gamma', \mathbb{C}) = 0 \quad \text{if } i \neq 0, d(G).$$

Thus theorem 2 immediately follows from theorem 1.

*Remark 6.* Garland's theorem is proven in the case of a single  $v$  but the proof easily extends to the case under consideration. □

**6.2. Theorem 8 and Simpson's integrality conjecture.** Notice that the conclusion of theorem 8 is predicted for  $d(G) \geq 2$  by Simpson's integrality conjecture [27] saying that any rigid representation of a Kähler group should be integral.

Indeed for simplicity let us assume that  $S = \{v\}$  and  $\mathbf{G}_v$  is of adjoint type. Fix an embedding  $F_v \xrightarrow{\sigma} \mathbb{C}$  and consider the natural faithful representation  $\rho : \Gamma \longrightarrow \mathbf{G}_v^\sigma(\mathbb{C})$  (where  $\mathbf{G}_v^\sigma := \mathbf{G}_v \times_{F_v, \sigma} \mathbb{C}$ ). As  $H^1(\Gamma, \text{Ad}\rho) = 0$  the representation  $\rho$  is rigid.

If  $\Gamma$  were a Kähler group then by Simpson's integrality conjecture there would exist a number field  $K \xrightarrow{\tau_0} \mathbb{C}$  and a  $K$ -form  $\mathbf{G}$  of  $\mathbf{G}_v^\sigma$  such that  $\rho(\Gamma) \subset \tau_0(\mathbf{G}(\mathcal{O}_K))$ .

By Margulis's superrigidity theorem [21], for any  $\tau$  in the finite set  $S_\infty$  of Archimedean places of  $K$  the representation  $\tau \circ \rho : \Gamma \longrightarrow \mathbf{G}^\tau(\mathbb{C})$  has bounded image : there exists a maximal compact subgroup  $U_\tau$  of  $\mathbf{G}^\tau(\mathbb{C})$  such that  $\tau(\rho(\Gamma)) \subset U_\tau$ . Thus

$$\rho(\Gamma) \subset \mathbf{G}(\mathcal{O}_K) \cap \prod_{\tau \in S_\infty} U_\tau .$$

However the intersection  $\mathbf{G}(\mathcal{O}_K) \cap \prod_{\tau \in S_\infty} U_\tau$  is finite as  $\mathbf{G}(\mathcal{O}_K)$  is discrete in  $\prod_{\tau \in S_\infty} \mathbf{G}^\tau(\mathbb{C})$ . Contradiction to the fact that the lattice  $\Gamma$  of  $G_v$  is infinite.

#### APPENDIX A. WHAT DOES CONJECTURE 1 MEAN ?

**A.1. In terms of group extensions.** Recall that for  $\Gamma$  a group and  $A$  an Abelian group with trivial  $\Gamma$ -module structure the group  $H^2(\Gamma, A)$  classifies the central  $A$ -extensions

$$0 \longrightarrow A \longrightarrow \tilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

of  $\Gamma$ . As  $b_2(\Gamma) = \text{rk} H^2(\Gamma, \mathbb{Z})$  the conjecture 1 means that any infinite Kähler group admits (after maybe passing to a finite index subgroup) a non-trivial  $\mathbb{Z}$ -central extension (which does not trivialize when restricted to any finite index subgroup).

**A.2. In topological terms.** For any group  $\Gamma$  the universal coefficients exact sequence yields the isomorphism

$$H^2(\Gamma, \mathbb{R}) = \text{Hom}_{\mathbb{R}}(H_2(\Gamma, \mathbb{R}), \mathbb{R}) .$$

In particular it is equivalent to show  $b_2(\Gamma) > 0$  or  $b^2(\Gamma) > 0$ . For any reasonable topological space  $M$  with fundamental group  $\Gamma$  the universal cover  $\tilde{M}$  is a principal  $\Gamma$ -cover of  $M$ . Thus it defines (uniquely in the homotopy category) a morphism  $c : M \longrightarrow B\Gamma$  from  $M$  to the classifying

Eilenberg-MacLane space  $B\Gamma = K(\Gamma, 1)$ . The induced morphism  $c_* : H_*(M, \mathbf{R}) \longrightarrow H_*(\Gamma, \mathbf{R})$  is easily seen to be an isomorphism in degree 1 and an epimorphism in degree 2 :

$$H_2(M, \mathbf{R}) \twoheadrightarrow H_2(\Gamma, \mathbf{R}) .$$

Dually :

$$H^2(\Gamma, \mathbf{R}) \hookrightarrow H^2(M, \mathbf{R}) .$$

How can we characterize the quotient  $H_2(\Gamma, \mathbf{R})$  of  $H_2(M, \mathbf{R})$  ? In fact this quotient first appeared in Hopf's work on the Hurewicz morphism comparing homotopy and homology :

**Theorem 9** (Hopf). *Let  $N$  be a paracompact topological space. Let  $c : N \longrightarrow B\pi_1(N)$  be the classifying morphism and  $h : \pi_*(N) \longrightarrow H_*(N, \mathbb{Z})$  the classical Hurewicz morphism. Then the sequence of Abelian groups*

$$(11) \quad \pi_2(N) \xrightarrow{h} H_2(N, \mathbb{Z}) \xrightarrow{c_*} H_2(\pi_1(N), \mathbb{Z}) \longrightarrow 0$$

*is exact.*

Cohomologically :

**Corollary 2.** *Let  $N$  be a paracompact topological space and  $\pi_2(N) \otimes_{\mathbb{Z}} \mathbf{R} \xrightarrow{h} H_2(N, \mathbf{R})$  the Hurewicz morphism. Then :*

$$H^2(\pi_1(N), \mathbf{R}) = \{[\omega] \in H^2(N, \mathbf{R}) / \forall \phi : S^2 \longrightarrow N, \langle [\omega], \phi_*[S^2] \rangle = 0\} \subset H^2(N, \mathbf{R}) ,$$

*where  $\langle \cdot, \cdot \rangle : H^2(N, \mathbf{R}) \times H_2(N, \mathbf{R}) \longrightarrow \mathbf{R}$  is the natural non-degenerate pairing between homology and cohomology.*

*Remark 7.* Nowadays theorem 9 is a direct application of the Leray-Cartan spectral sequence.

Carlson-Toledo's conjecture can thus be restated :

**Conjecture 3.** *Let  $\Gamma$  be an infinite Kähler group. There exists a compact Kähler manifold  $M$  with  $\pi_1(M)$  a finite index subgroup of  $\Gamma$  such that the Hurewicz morphism  $\pi_2(M) \otimes_{\mathbb{Z}} \mathbf{R} \longrightarrow H_2(M, \mathbf{R})$  is not surjective.*

A stronger conjecture is then :

**Conjecture 4.** *Let  $M$  be a compact Kähler manifold with infinite fundamental group  $\Gamma$ . There exists a finite étale cover  $M'$  of  $M$  such that the Hurewicz morphism  $\pi_2(M) \otimes_{\mathbb{Z}} \mathbf{R} \longrightarrow H_2(M', \mathbf{R})$  is not surjective.*

Of course  $\pi_2(M)$  is nothing else than  $H_2(\tilde{M}, \mathbb{Z})$  where  $\tilde{M}$  denotes the universal cover of  $M$ . The cohomological version of the previous conjecture gives :

**Conjecture 5.** *Let  $M$  be a compact Kähler manifold with infinite fundamental group  $\Gamma$ . Then the natural map*

$$\varinjlim_{M'} H^2(M', \mathbf{R}) \longrightarrow H^2(\tilde{M}, \mathbf{R})$$

*is not injective (where the inductive limit is taken over the projective system of étale finite cover of  $M$ ).*

Notice that for any compact manifold  $M$  and any finite étale cover  $M'$  of  $M$  the arrow  $H^2(M, \mathbf{R}) \longrightarrow H^2(M', \mathbf{R})$  is injective by the projection formula.

**A.3. In terms of  $\mathbb{C}^*$ -bundles.** Recall that for a reasonable topological space  $M$  the group  $H^2(M, \mathbb{Z})$  canonically identifies with the group  $\mathcal{L}(M)$  of principal  $\mathbb{C}^*$ -bundles : on the one hand  $H^2(M, \mathbb{Z}) = [M, K(\mathbb{Z}, 2)]$ , on the other hand  $\mathcal{L}(M) = [M, B\mathbb{C}^*]$ . But both  $K(\mathbb{Z}, 2)$  and  $B\mathbb{C}^*$  have as canonical model the infinite projective space  $\mathbb{CP}^\infty$ . Thus conjecture 1 states that any infinite Kähler group  $\Gamma$  admits a finite index subgroup  $\Gamma'$  whose classifying space  $B\Gamma'$  supports a non-trivial  $\mathbb{C}^*$ -torsor. Let  $M$  be a compact Kähler manifold with infinite fundamental group  $\Gamma$ . As  $\tilde{M}$  is the homotopy fiber of  $M \rightarrow B\Gamma$  the conjecture 5 says there exists a finite étale cover  $M'$  of  $M$  and a non-trivial  $\mathbb{C}^*$ -torsor  $L'$  on  $M'$  whose pull-back to  $\tilde{M}$  becomes trivial. In these statements “non-trivial” means “with non-trivial *rational* first Chern class”.

**A.4. Equivalence.** The equivalence of these 3 points of view is clear. Given  $M$  and a  $\mathbb{C}^*$ -torsor  $L$  on  $M$  one can consider the long homotopy exact sequence for the fibration  $L \rightarrow M$  :

$$(12) \quad \cdots \rightarrow \pi_2(L) \rightarrow \pi_2(M) \xrightarrow{c_1} \pi_1(\mathbb{C}^*) = \mathbb{Z} \rightarrow \pi_1(L) \rightarrow \pi_1(M) \rightarrow 1 .$$

The boundary map  $c_1 : \pi_2(M) \rightarrow \mathbb{Z}$  is just the first Chern class map of  $L$  restricted to  $\pi_2(M)$  : if  $[\alpha] \in \pi_2(M)$  is represented by  $\alpha : S^2 \rightarrow M$  then  $c_1([\alpha]) = \langle \alpha^*(c_1(L)), [S^2] \rangle \in \mathbb{Z}$ . As  $H_2(M, \mathbb{Z}) = \pi_2(M)$  and  $H^2(\tilde{M}, \mathbb{R})$  is dual to  $H_2(M, \mathbb{R})$  the torsor  $p^*(L)$  is trivial if and only if  $c_1 : \pi_2(M) \rightarrow \mathbb{R}$  is zero. Then the long exact sequence (12) gives the short exact sequence :

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(L) \rightarrow \pi_1(M) \rightarrow 1 .$$

#### APPENDIX B. CARLSON-TOLEDO’S CONJECTURE AND RIGIDITY

The strongest evidence for Carlson-Toledo’s conjecture is the following folkloric theorem, essentially due to Lefschetz, Simpson and Reznikov (a proof is provided in [17]) :

**Theorem 10.** *Let  $\Gamma$  be a Kähler group. If  $\Gamma$  admits a linear representation  $\rho : \Gamma \rightarrow G$ , with  $G$  the linear group  $GL(V)$  of a finite dimensional complex vector space  $V$  or the isometry group of a Hilbert space, satisfying  $H^1(\Gamma, \rho) \neq 0$ , then  $b_2(\Gamma) > 0$ .*

Recall the following definitions :

**Definition 5.** *A finitely generated group  $\Gamma$  is said*

- (1) *rigid if for any  $n \in \mathbb{N}$   $\Gamma$  admits only finitely many simple linear representations of dimension  $n$  (up to conjugacy).*
- (2) *reductive if all its linear finite-dimensional complex representations are semi-simple.*
- (3) *schematically rigid if for any  $n \in \mathbb{N}$  and any representation  $\rho : \Gamma \rightarrow \mathbf{GL}(n, \mathbb{C})$  one has  $H^1(\Gamma, \text{Ad}\rho) = 0$ .*

*Remark 8.* In [20] a rigid group is called *S-rigid* (*SS* for semi-simple).

Let  $A(\Gamma)$  be the pro-(affine)-algebraic completion of  $\Gamma$ . For  $n \in \mathbb{N}$  let  $A_n(\Gamma)$  be the quotient  $A(\Gamma)/K_n(\Gamma)$  where  $K_n(\Gamma)$  denotes the intersection of the kernels of all representations  $A(\Gamma \rightarrow GL(n, \mathbb{C}))$ . All  $\Gamma$ -representations of dimension  $n$  factorizes uniquely through  $A_n(\Gamma)$ . One easily shows that  $\Gamma$  is rigid if and only if for all  $n \in \mathbb{N}$  the group  $A_n(\Gamma)$  is an affine algebraic group (i.e. of finite dimension) (c.f. [3, theorem A]).

In the definition 5 each condition implies the previous one :

As the  $\mathbf{GL}(n, \mathbb{C})$ -orbit of a semi-simple representation  $\rho \in \text{Hom}(\Gamma, \mathbf{GL}(n, \mathbb{C}))$  is closed condition (2) implies condition (1). On the other hand there exists non-reductive rigid groups : for example  $\mathbf{SL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ ,  $n \geq 3$ .

Any schematically-rigid group is rigid of course. As the  $GL(n, \mathbb{C})$ -orbit in  $\text{Hom}(\Gamma, \mathbf{GL}(n, \mathbb{C}))$  of any representation contains a semi-simple representation in its closure any schematically-rigid group is reductive. To say that  $\Gamma$  is schematically rigid is equivalent to saying that  $A(\Gamma)$  is pro-reductive. There exist rigid groups which are not schematically rigid : c.f. [3, section 5].

As a corollary of theorem 10 one obtains :

**Corollary 3.** *Let  $\Gamma$  an infinite Kähler group. If  $\Gamma$  does not satisfy Carlson-Toledo's conjecture then necessarily :*

- (a)  $\Gamma$  has Kazhdan's property (T).
- (b)  $\Gamma$  is schematically rigid.

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